Math 2 Unit 8: Probability

Approximate Time Frame: 3 – 4 Weeks

Connections to Previous Learning:
Students will build upon their understanding of basic probability (chance, models, and sample spaces, relative frequency) to gain skill in independent and conditional probability.

Focus of this Unit:
Students will understand independence and conditional probability and use them to interpret data. Students will be able to compute probabilities of independent, dependent, and compound events.

Connections to Subsequent Learning:
Students will experience the strong connection between statistics and probability using data to select values for probability models.

From the High School Statistics and Probability Progression Document pp. 8, 13-17

Making inferences and justifying conclusions:
Students now move beyond analyzing data to making sound statistical decisions based on probability models. The reasoning process is as follows: develop a statistical question in the form of a hypothesis (supposition) about a population parameter; choose a probability model for collecting data relevant to that parameter; collect data; compare the results seen in the data with what is expected under the hypothesis. If the observed results are far away from what is expected and have a low probability of occurring under the hypothesis, then the hypothesis is called into question. In other words, the evidence against the hypothesis is weighed by the probability.

Conditional probability and the rules of Probability: In Grades 7 and 8, students encountered the development of basic probability, including chance processes, probability models, and sample spaces. In high school, the relative frequency approach to probability is extended to conditional probability and independence, rules of probability and their use in finding probabilities of compound events, and the use of probability distributions to solve problems involving expected value. As seen in the making inferences section above, there is a strong connection between statistics and probability. This will be seen again in this section with the use of data in selecting values for probability models.

Possible outcomes: Guessing on four true–false questions

Number correct	Out-comes	Number correct	Out-comes	Number correct	Out-comes
4	CCCC	2	CCII	1	CIII
3	ICCC	2	CICI	1	ICII
3	CICC	2	CIIC	1	IICI
3	CCIC	2	ICCI	1	IIIC
3	CCCI	2	ICIC	0	IIII
		2	IICC		

C indicates a correct answer; I indicates an incorrect answer.

Understand independence and conditional probability and use them to interpret data: In developing their understanding of conditional probability and independence, students should see two types of problems, one in which the uniform probabilities attached to outcomes leads to independence and one in which it does not. For example, suppose a student is randomly guessing the answers to all four true–false questions on a quiz. The outcomes in the sample space can be arranged as shown in the margin. Probabilities assigned to these outcomes should be equal because random guessing implies that no one outcome should be any more likely than another.
By simply counting equally likely outcomes:

$$\text{P(exactly two correct answers)}=\frac{6}{16} \text{ and P(at least one correct answer)}=\frac{15}{16}$$
$$=1\text{-P (no correct answers)}$$

Likewise, P(C on first question) $= \frac{1}{2}$ as should seem intuitively reasonable. Now, P[(C on first question) and (C on second question)] $= \frac{4}{16}$

$$= \frac{1}{4}$$

$$= \frac{1}{2} \cdot \frac{1}{2}$$

which shows that the two events (C on first question) and (C on second question) are independent, by the definition of independence. This, too, should see intuitively reasonable to students because the random guess on the second question should not have been influenced by the random guess on the first.

Students may contrast the quiz scenario above with the scenario of choosing at random two students to be leaders of a five-person working group consisting of three girls (April, Briana, and Cyndi) and two boys (Daniel and Ernesto). The first name chosen indicates the discussion leader and the second the recorder, so order of selection is important. The 20 outcomes are displayed in the margin.

Selecting two students from three girls and two boys

Number of girls	Outcomes	
2	AB	BA
2	AC	CA
2	BC	CB
1	AD	DA
1	AE	EA
1	BD	DB
1	BE	EB
1	CD	DC
1	CE	EC
0	DE	ED

Here, the probability of selecting two girls is:

P(two girls selected) $= \frac{6}{20} = \frac{3}{10}$ whereas P(girl selected on first draw) $= \frac{12}{20} = \frac{3}{5}$ = P (girl selected on second draw)

Because $\frac{3}{5} \cdot \frac{3}{5} \neq \frac{3}{10}$, these two events are not independent. The selection of the second person does depend on the selection of the first when the same person cannot be selected twice.

Another way of viewing independence is to consider the conditional probability of an event A given an event B, P(A|B), as the probability of A in the sample space restricted to just those outcomes that constitute B. In the table of outcomes for guessing on the true-false questions,

$$\text{P(C on second question | C on the first question)} = \frac{4}{8} = \frac{1}{2}$$

and students see that knowledge of what happened on the first question does not alter the probability of the outcome on the second; the two events are independent.

HIV risk by age groups, in percent of population

	Age	18–24	25–44	45–64	Row Total
Not at risk	Row %	14.0	59.6	26.4	100.0
	Column %	35.0	51.7	27.2	
	Total %	5.6	23.6	10.5	39.6
At risk	Row %	17.1	36.5	46.4	100.0
	Column %	65.0	48.3	72.8	
	Total %	10.3	22.0	28.1	60.4
Column total	Row %	15.9	45.6	38.5	100.0
	Column %	100.0	100.0	100.0	100.0
	Total %	15.9	45.6	38.5	100.0

Source: Center for Disease Control,
http://apps.nccd.cdc.gov/s_broker/WEATSQL.exe/weat/freq_year.hsql

In the selecting students scenario, the conditional probability of a girl on the second selection, given that a girl was selected on the first is:

$$\text{P(girl on the second | girl on first)} = \frac{6}{12} \text{ and P(girl on second)} = \frac{3}{5}$$

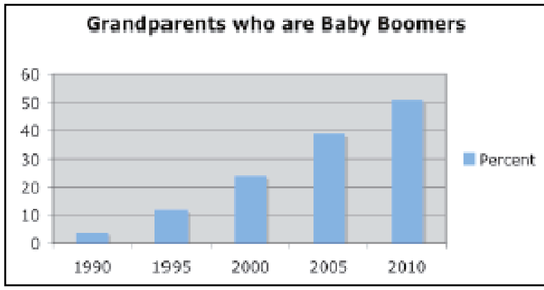
$$= \frac{1}{2}$$

So, these two events are again seen to be dependent. The outcome of the second draw does depend on what happened at the first draw.

Students understand that in real world applications the probabilities of events are often approximated by data about those events. For example, the percentages in the table for HIV risk by age group (p. 4) can be used to approximate probabilities of HIV risk with respect to age or age with respect to HIV risk for a randomly selected adult from the U.S. population of adults. Emphasizing the conditional nature of the row and column percentages:
$\text{P(adult is age 18 to 24} \mid \text{adult is at risk)} = 0.171$ whereas $\text{P(adult is at risk} \mid \text{adult is age 18 to 24)} = 0.650$

Comparing the latter to $\text{P(adult is at risk} \mid \text{adult is age 25 to 44)} = 0.483$ shows that the conditional distributions change from column to column, reflecting dependence and an association between age category and HIV risk.

Students can gain practice in interpreting percentages and using them as approximate probabilities from study data presented in the popular press. Quite often the presentations are a little confusing and can be interpreted in more than one way. For example, two data summaries from *USA Today* are shown below. What might these percentages represent and how might they be used as approximate probabilities?



Top age groups for DUI	
21–25	29%
26–29	24%
18–20	20%
30–34	19%

Use the rules of probability to compute probabilities of compound events in a uniform probability model: The two-way table for HIV risk by age group (p. 4) gives percentages from a data analysis that can be used to approximate probabilities, but students realize that such tables can be developed from theoretical probability models. Suppose, for example, two fair six-sided number cubes are rolled, giving rise to 36 equally likely outcomes.

Outcomes for specified events can be diagramed as sections of the table, and probabilities calculated by simply counting outcomes. This type of example is one way to review information on conditional probability and introduce the addition and multiplication rules. For example, defining events:

A is "you roll numbers summing to 8 or more" B is "you roll doubles" and counting outcomes leads to

P(A) = $\frac{15}{36}$

P(B) = $\frac{6}{36}$

P(A and B) = $\frac{3}{16}$ and P(BIA) = $\frac{3}{15}$, the fraction of A's 15 outcomes that also fall in B.

Now, by counting outcomes: P(A or B) = $\frac{18}{36}$

or by using the Addition Rule P(A or B) = P(A) + P(B) – P(A and B)

$$= \frac{15}{36} + \frac{6}{36} - \frac{3}{36}$$

$$= \frac{18}{36}$$

Possible outcomes: Rolling two number cubes

	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Or by using the Multiplication Rule

P(A and B) = P(A)P(BIA)

$$= \frac{15}{36} \cdot \frac{3}{15}$$

$$= \frac{3}{36}$$

The assumption that all outcomes of rolling each cube once are equally likely results in the outcome of rolling one cube being independent outcome of rolling the other. Students should see that independence is often used as a simplifying assumption in constructing theoretical probability models that approximate real situations. Suppose a school laboratory has two smoke alarms as a built in redundancy for safety. One has probability 0.4 of going off when steam (not smoke) is produced by running hot water and the other has probability 0.3 for the same event. The probability that they both go off the next time someone runs hot water in the sink can be reasonably approximated as the product $0.4 \cdot 0.3 = 0.12$ even though there may be some dependence between two systems operating in the same room. Modeling independence is much easier than modeling dependence, but models that assume independence are still quite useful.

Desired Outcomes

Standard(s):

Understand independence and conditional probability and use them to interpret data.

- S.CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions intersections, or complements of other events (or, and, not).
- S.CP.2 Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.
- S.CP.3 Understand the conditional probability of A given B as P (A and B)/P(B), and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.
- S.CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*
- S.CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.

Use the rule of probability to compute probabilities of compound events in a uniform probability model.

- S.CP.6 Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.
- S.CP.7 Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model.

Transfer:

Students will use concepts and procedures to find probabilities in various situations.

Ex. The Pythagorean Society needs to form a 5 member task force from its membership of 25 students, which includes the twins Soh Yung and Tou Yung. What is the probability that a task force is formed that includes:

a) Neither of the twins?

b) Both of the twins?

Ex. Auntie Emsaw is about to fix 6 eggs for breakfast. In the refrigerator, there are 21 good eggs and 3 rotten ones. What is the probability that she will choose the 6 eggs so that

a) exactly one of the eggs is rotten?

b) exactly one of the eggs is rotten given that she knows the first two eggs she chose were good?

c) at least one of the eggs is rotten?

WIDA Standard: (English Language Learners)

English language learners communicate information, ideas and concepts necessary for academic success in the content area of Mathematics.
English language learners benefit from:

- Hands-on activities and visual models for developing and analyzing probability models.
- Explicit vocabulary instruction in probability language and contexts.

Understandings: *Students will understand that…*

- Events can be described as a subset of a sample space.
- The probability of two events occurring together is the product of their probabilities, if and only if then the events are independent.
- The probability of two events can be conditional on each other and the interpretation of that probability.
- Two-way frequency tables can be used to decide if events are independent and to find conditional probabilities.
- Conditional probability and independence are applied to everyday situations.
- Conditional probability of A given B can be found and interpreted in context.
- The addition rule can be applied and the resulting probability can be interpreted in context.

Essential Questions:

- How can an event be described as a subset of outcomes using correct set notation?
- How are probabilities, including joint probabilities, of independent events calculated?
- How are probabilities of independent events compared to their joint probability?
- How does conditional probability apply to real-life events?
- How are two-way frequency tables used to model real-life data?
- How are conditional probabilities and independence interpreted in relation to a situation?
- What is the difference between compound and conditional probabilities?
- How is the probability of event (A or B) found?

Mathematical Practices: (Practices to be explicitly emphasized are indicated with an *.)

* 1. **Make sense of problems and persevere in solving them.** Students will understand the context of a problem and use this understanding to work to the solution.

* 2. **Reason abstractly and quantitatively.** Students will be able to move from the use of a formula to the application to the context of a problem.

* 3. **Construct viable arguments and critique the reasoning of others.** Students will be able to defend their results and conclusions, as well as comparing and contrasting their results with the results of other students.

* 4. **Model with mathematics.** Students use two-way frequency tables as they study dependent and independent variables in real-world probability contexts.

5. **Use appropriate tools strategically.**

* 6. **Attend to precision.** Students will interpret the possible number of solutions to a problem.

7. **Look for and make use of structure.**

8. **Look for and express regularity in repeated reasoning.**

Prerequisite Skills/Concepts:

Students should already be able to:

- Represent sample spaces.
- Apply basic properties of probability.
- Use Venn diagrams and two-way frequency tables.
- Use $P(A \cap B)$ as the probability of A and B occurring together.

Advanced Skills/Concepts:

Some students may be ready to:

- Pose an original question, prepare a solution, and interpret the result.
- Investigate the relationships between $P(A \cap B)$, $P(A \cup B)$, $P(A | B)$, and $P(A | B)$ using tree diagrams and Bayes Theorem.

Knowledge: *Students will know…*	**Skills:** *Students will be able to …*
• The definition of event, sample space, union, intersection, and complement. • How to identify independent events. • The definition of dependent events and conditional probability.	• Establish events as subsets of sample space based on the union, intersection, and/or complement of other events. • Calculate the probability of an event. • Determine if two events are independent with justification. • Calculate the conditional probability of A given B. • Use the concept of conditional probability and independence using real life examples. • Calculate the probability of the intersection of two events. • Calculate the conditional probability of A given B. • Determine the probability of the union of two events using the Addition Rule.

Academic Vocabulary:

Critical Terms:	**Supplemental Terms:**
Joint probability Event Independent events Conditional Conditional probability Independence Marginal probability Random variable	Sample space Subset Outcome Union Intersection Complement Set notation

Assessments

Pre-Assessments	Formative Assessments	Summative Assessments	Self-Assessments
	#1 Defining the Sample #3 Dealing with Probability #4 Are They Independent #6 Probability Potter	#2 Adding It Up #5 Hunger Games Probability$^{\mu}$ #7 Frequency Tables	

Sample Lesson Sequence
1. S.CP.1, S.CP.7 Investigating Subsets, Sample space, and the addition rule. **2. S.CP.6, S.CP.2, S.CP.3, S.CP.5 Learn about Conditional probability and using it to show independence.**