Math 2 Unit 3 Quadratic Functions: Working with Equations

Approximate Time Frame: 4 – 5 Weeks

Connections to Previous Learning:

In the previous unit, students learned to represent quadratic functions algebraically, in tables, and graphically and to analyze these functions. They also learned to rewrite quadratics in different forms through completing the square and factoring. In this unit, students will extend this knowledge and their understanding of solving equations from Math 1 to solving and interpreting the solution of quadratic equations.

Focus of this Unit:

In this unit, students will learn how to solve quadratic equations, including those with complex solutions, and to relate the solution of a quadratic equation to the zeros of a quadratic function.

Connections to Subsequent Learning:

Students will build upon the solution methods learned here when modeling quadratic functions in the next unit and in their work with polynomial and rational functions in Math 3.

From the High School, Algebra Progression Document, pp. 10-11:

Equations in one variable

A naked equation, such as $x^2 = 4$, without any surrounding text, is merely a sentence fragment, neither true nor false, since it contains a variable x about which nothing is said. A written sequence of steps to solve an equation, such as in the margin, is code for a narrative line of reasoning using words like "if", "then", "for all" and "there exists." In the process of learning to solve equations, students learn certain standard "if-then" moves, for example if x = y then x + 2 = y + 2." The danger in learning algebra is that students emerge with nothing but the moves, which may make it difficult to detect incorrect or made-up moves later on. Thus the first requirement in the standards in this domain is that students understand that solving equations is a process of reasoning. This does not necessarily mean that they always write out the full text; part of the advantage of algebraic notation is its compactness. Once students know what the code stands for, they can start writing in code. Thus, eventually students might make $x^2 = 4 \rightarrow x = \pm 2$ one step.

Understanding solving equations as a process of reasoning demystifies "extraneous" solutions that can arise under certain solution procedures. The flow of reasoning is forward, from the assumption that a number x satisfies the equation to a list of possibilities for x. But not all the steps are necessarily reversible, and so it is not necessarily true that every number in the list satisfies the equation. For example, it is true that if x = 2 then $x^2 = 4$. But it is not true that if $x^2 = 4$ then x = 2 (it might be that x = -2). Squaring both sides of an equation is a typical example of an irreversible step; another is multiplying both sides of the equation by a quantity that might be zero (see examples).

Fragments of reasoning

$$\begin{aligned} x^2 &= 4 \\ x^2 - 4 &= 0 \\ (x-2)(x+2) &= 0 \\ x &= 2, -2 \end{aligned}$$

This sequence of equations is short-hand for a line of reasoning: "If x is a number whose square is 4, then $x^2 - 4 = 0$. Since $x^2 - 4 = (x-2)(x+2)$ for all numbers x, it follows that $(x-2)(x+2) = 0$. So either $x - 2 = 0$, in which case $x = 2$, or $x + 2 = 0$, in which case $x = -2$." More might be said: a justification of the last step, for example, or a check that 2 and -2 actually do satisfy the equation, which has not been proved by this line of reasoning.

Major Standards = Approximately 70% Supporting Standards = Approximately 20% Additional Standards = Approximately 10%

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With an understanding of solving equations as a reasoning process, students can organize the various methods for solving different types of equations into a coherent picture. For example, solving linear equations involves only steps that are reversible (adding a constant to both sides, multiplying both sides by a non-zero constant, transforming an expression on one side into an equivalent expression). Therefore solving linear equations does not produce extraneous solutions. The process of completing the square also involves only this same list of steps, and so converts any quadratic equation into an equivalent equation of the form $(x\text{-}p)^2$ that has exactly the same solutions solve by the reasoning explained above.

This example sets up a theme that reoccurs throughout algebra; finding ways of transforming equations into certain standard forms that have the same solutions. For example, any exponential equation can be transformed into the form $b^x = a$, the solution to which is (by definition) a logarithm.

It is traditional for students to spend a lot of time on various techniques of solving quadratic equations, which are often presented as if they are completely unrelated (factoring, completing the square, the quadratic formula). In fact, as we have seen, the key step in completing the square, going from $x^2 = q$ to $x = \pm\sqrt{q}$, involves at its heart factoring. And the quadratic formula is nothing more than an encapsulation of the method of completing the square. Rather than long drills on techniques of dubious value, students with an understanding of the underlying reasoning behind all these methods are opportunistic in their application, choosing the method that bests suits the situation at hand.

Desired Outcomes

Standard(s):

Perform arithmetic operations with complex numbers.

- N.CN.1 Know there is a complex number *I* such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
- N.CN.2 Use the relation $i^2 = -1$ and the commutative, associate, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

- N.CN.7 Solve quadratic equations with real coefficients that have complex solutions.

Reason quantitatively and use units to solve problems.

- N.Q.2 Define appropriate quantities for the purpose of descriptive modeling.

Create equations that describe numbers or relationships.

- A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*

Interpret the structure of expressions.

- A.SSE.1 Interpret expressions that represent a quantity in terms of its context.
  b) Interpret complicated expressions by viewing one or more of their parts as a single entity. ~~*For example, interpret P(1+r)n as the product of P and a factor not depending on P.*~~
- A.SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares*

that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

Write expressions in equivalent forms to solve problems.

- A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
  a) Factors a quadratic expression to reveal the zeros of the function it defines.
  b) Complete the square in a quadratic expression to reveal the maximum value of the function it defines.

Understand solving equations as a process of reasoning and explain the reasoning.

- A.REI.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

- A.REI.4 Solve quadratic equations in one variable.
  a) Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
  b) Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations.

- A.REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ and the circle $x^2 + y^2 = 3$.

Transfer:

Students will use quadratic functions to model situations, use various parts of the function to determine information about the model, and define appropriate quantities when modeling.

Ex. The function $h = -16t^2 + 2300$ gives an object's height, h, in feet, at t seconds.

- What does the constant 2300 tell you about the height of the object?
- What does the coefficient of t^2 tell you about the direction the object is moving?
- What are a reasonable domain and range for the function h?

WIDA Standard: (English Language Learners)

English language learners communicate information, ideas and concepts necessary for academic success in the content area of Mathematics.

English language learners benefit from:

- the use of tactile and virtual manipulatives for studying the process of completing the square.
- explicit vocabulary instruction with regard to the components of equations and graphs of quadratic functions.
- guided discussions regarding the connections between real-world situations, graphs, tables and equations.

Understandings: *Students will understand that…*

- Applied problems using quadratics can be answered by either solving a quadratic equation or re-writing the quadratic in a more useful form (factoring to find the zeros, or completing the square to find the maximum or minimum, for instance).
- There are several ways to solve a quadratic equation (square roots, completing the square, quadratic formula, and factoring), and that the most efficient route to solving can often be determined by the initial form of the equation.
- The quadratic formula is derived from the process of completing the square.
- Non-real numbers exist and can arise in the solutions of quadratic equations.
- A quadratic function that does not intersect the x-axis has complex zeros.
- The relationship between the factors of a quadratic and the x-intercepts of the graph of the quadratic.

Essential Questions:

- How can a quadratic equation be solved?
- What are complex numbers, and why do they exist?
- How is the quadratic formula derived?
- How do the factors of a quadratic determine the x-intercepts of the graph and vice versa?

Mathematical Practices: (Practices to be explicitly emphasized are indicated with an *.)

1. **Make sense of problems and persevere in solving them.**

*2. **Reason abstractly and quantitatively.** Students demonstrate this practice when they can translate back and forth between the representation of a quadratic and its applied meaning (for instance, completing the square on a quadratic describing projectile motion to find its vertex and then interpreting the result as the maximum height the projectile obtained, and then setting the quadratic equal to zero and solving to find the time at which the projectile reached the ground, potentially describing why the negative solution is not viable).

3. **Construct viable arguments and critique the reasoning of others.**

*4. **Model with mathematics**. Students can use quadratic models to describe applied situations, such as projectile motion or functions describing revenue or profit (for instance, the function $R = c(2000-10c)$ might describe the revenue generated by setting the price of a product at c, where $2000-10c$ would be the number of units sold based on the price).

*5. **Use appropriate tools strategically.** Students will use a graphing tool to explore the relationship between the factors of a quadratic and the zeros of its graph.

6. **Attend to precision.**

*7. **Look for and make use of structure.** Students will continue to look for and make use of structure as they factor, complete the square, and use the quadratic formula to solve quadratics or highlight different quantities of interest.

8. **Look for and express regularity in repeated reasoning**.

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Prerequisite Skills/Concepts:	Advanced Skills/Concepts:
Students should already be able to: • Factor quadratics. • Complete the square on quadratics. • Solve linear and exponential equations. • Use properties of rational and irrational numbers to represent and solve equations.	*Some students may be ready to:* • Graph complex numbers. • Determine the magnitude of complex numbers. • Recognize and use patters in powers of *i*. • Factor quadratics over the complex numbers. • Find the conjugate of a complex number. • Find products and quotients of complex numbers. • Recognize that if a complex number is a solution to a given quadratic, so is its conjugate. • N.CN.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers. • N.CN.8 (+) Extend polynomial identities to the complex numbers. For example, rewrite $x^2 + 4$ as $(x + 2i)(x – 2i)$. • N.CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.
Knowledge: *Students will know…* • That there is a number i such that $i^2 = -1$. • That every complex number can be written in the form $a + bi$, where a and b are real numbers. • The terms root, zero, and x-intercept are synonymous.	**Skills**: *Students will be able to …* • Perform arithmetic operations on complex numbers. • Interpret complicated expressions by viewing one or more of their parts as a single entity. • Define appropriate quantities when modeling. • Explain their reasoning in solving equations. • Solve quadratic equations by taking square roots. • Solve quadratic equations by completing the square. • Solve quadratic equations by factoring. • Solve quadratic equations using the quadratic formula. • Derive the quadratic formula by completing the square.

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Academic Vocabulary:

Critical Terms:	Supplemental Terms:
Complex Number Discriminant Factor Zero Root x-intercept	Maximum Minimum Vertex

Assessments

Pre-Assessments	Formative Assessments	Summative Assessments	Self-Assessments
0 Unit 3 Pre-Assessment	*1 Solving Quadratic Equations* *3 Using Quadratic Formula* *4 Complex Numbers and Solving Quadratics* *5 Solving Quadratics Puzzle* *6 Basketball Task*	*3 Using Quadratic Formula* *1 Solving Quadratic Equations* *4 Complex Numbers and Solving Quadratics* *6 Basketball Task*	*0 Unit 3 Pre-Assessment*

Sample Lesson Sequence:

1. **A.SSE.1, 2, 3 N.Q.2 Solving quadratic equations algebraically – Sample Lesson Plan (5 days)**
   a. **Solving quadratic equations using factoring**
   b. **Solving quadratic equations using the properties of square roots**
   c. **Solving quadratic equations by completing the square**
2. A.REI.1, 4, 7, A.CED. 4 Using the process of completing the square with a quadratic equation in standard form to derive the Quadratic formula. 6 days
   a. Noticing patterns when solving quadratic equations using completing the square
   b. Derive the Quadratic formula
   c. Solve problems using the quadratic formula
3. N.CN.1, 2, 7, A.SSE.3, N.Q.2 Complex numbers, including real and non-real numbers, as solutions of quadratic equations. 5 days
   a. Equations with a negative discriminate
   b. Complex numbers (real and non-real)
   c. Operations with complex numbers